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Abstract

We investigate the effect of *T*-duality on non-commutativity. Starting with open strings ending on a D2-brane wrapped on a T^2 torus in the presence of a Kalb–Ramond field, we consider Buscher transformations on the coordinates and background. We find that the dual system is commutative. We also study alternative transformations that can preserve non-commutativity.

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1. Introduction

T-duality [1–3] is one of the most interesting symmetries of string theory since it relates small-scale physics to large-scale physics. When one target space dimension is compact, the strings do not distinguish whether the compactification radius is R or α'/R and both possible worlds are related by *T*-duality. If d space coordinates are compactified in a torus T^d, there are several *T*-duality transformations associated with the conformal symmetries of the torus. These transformations act on the torus metric g_{ij} and also on the winding and momentum numbers of the compact coordinates in such a way that the Hamiltonian is preserved.

In the absence of an antisymmetric Kalb–Ramond field B_{ij}, some *T*-duality transformations can be realized as simple transformations of the string coordinates. If this field is turned on, considering the standard *T*-dualization procedure, the metric g_{ij} and coordinates X^i have a non-trivial transformation.

In the context of open string theory, the Kalb–Ramond field plays a crucial role because it may lead to non-commutativity at the string endpoints [4–7]. It is interesting to ask whether this non-commutativity is preserved or not after a *T*-duality transformation. A very important discussion of this problem can be found in [7], where open string background parameters were introduced. Another possible approach would be to analyze the effect of *T*-duality on the boundary conditions. D-branes on a non-commutative torus were studied in [8]. See also [9–12] for discussions of *T*-duality and non-commutativity.

The aim of this paper is to investigate the effect of *T*-duality transformation for open strings in non-commutativity. We consider as our primal system a D2-brane which wraps a T^2-torus in the presence of a constant Kalb–Ramond field with spatial components (magnetic field). This is a non-commutative system. We perform *T*-duality by considering background and coordinate transformations of the form proposed by Buscher [13]. Analyzing not only the background but also the boundary conditions, we find that the dual system has a *commutative* character. We discuss this result and the fact that the dual target space still allows non-commutativity, if one starts with different primal systems. Another discussion of non-commutativity and *T*-duality, for the case of a background electric field, can be found in [14].

Inspired in the case of a zero Kalb–Ramond field, we also consider an alternative transformation consisting of the interchange of the τ and σ derivatives. When applied to two coordinates, this transformation generates a dual D2-brane with a nonzero Kalb–Ramond field, preserving non-commutativity. We show that this transformation is a symmetry of the Hamiltonian but violates the condition that winding and momentum modes must be an integer for closed strings so it is not a *T*-duality.

We begin with a discussion, in section 2, of non-commutativity on a T^2 torus with the Kalb–Ramond background. Next, in section 3, we make a general discussion of *T*-duality in terms of transformations involving the background fields and the winding and momentum numbers, and then zoom in on the case in point: the D2-brane. In section 4, we study *T*-duality for the open string coordinates considering separately the Buscher dualization of one or two coordinates along the D2-brane. We also discuss the alternative transformation for these cases. The commutative/non-commutative character of the dual theories obtained is discussed. We follow through with the conclusions.

2. D2-brane in a torus with a constant Kalb–Ramond field: open strings non-commutativity

Consider a torus T^2 formed by two compact angular coordinates

$$X^i \sim X^i + 2\pi \tag{1}$$

with $i = 1, 2$. Using these angular coordinates the radii of the torus are inserted in the metric. We denote the non-compact coordinates as X^I with $I \neq 1, 2$. For simplicity, the only non-vanishing component of our Kalb–Ramond B field will be $B_{12} = -B_{21} =: \mathcal{B}$ where $\mathcal{B}$ is a constant. The toroidal contribution to the worldsheet action of an open string propagating in the presence of B_{ij} can be written as

$$S = \frac{1}{4\pi} \int d\tau \, d\sigma [-h^{\alpha\beta} g_{ij} \partial_\alpha X^i \partial_\beta X^j + \epsilon^{\alpha\beta} B_{ij} \partial_\alpha X^i \partial_\beta X^j], \tag{2}$$

where $h_{\alpha\beta} = \text{diag}(-, +)$ and metric g_{ij} is diagonal, with elements R_i^2/α', where R_i are the radii of the torus ($i = 1, 2$). The equations of motion are

$$\ddot{X}^i - X''^i = 0 \qquad i = 1, 2, \tag{3}$$

where $\dot{X}^i := \partial_\tau X^i$ and $X'^i := \partial_\sigma X^i$. The action of equation (2) differs from the free open string case by the Kalb–Ramond term which is a surface term that modifies the boundary conditions (BCs):

$$\delta X^i (g_{ij} X'^j + B_{ij} \dot{X}^j)|_{\sigma=0}^{\sigma=\pi} = 0 \tag{4}$$

and, consequently, the commutation relations $[X^i(\tau, \sigma), X^j(\tau, \sigma')]$ at the string endpoints, as we shall see. We will choose Dirichlet conditions for the non-compact coordinates X^I.

From equation (4), we see that we have two possible BCs for the open string coordinates X^1 and X^2 at the endpoints:

(i) Dirichlet conditions: $\delta X^i = 0$, or

(ii) Mixed conditions: $g_{ij} X'^j + B_{ij} \dot{X}^j = 0$.

The D2-brane corresponds to choosing mixed conditions for both X^1 and X^2 at $\sigma = 0, \pi$:

$$g_{11} X'^1 + \mathcal{B} \dot{X}^2 = 0, \qquad g_{22} X'^2 - \mathcal{B} \dot{X}^1 = 0. \tag{5}$$

The solutions for string coordinates X^1 and X^2 satisfying the equations of motion and mixed boundary conditions have the following form:

$$X^1(\tau, \sigma) = x^1 + w_1 \tau - \frac{\mathcal{B}}{g_{11}} w_2 \sigma + \frac{i}{n} \alpha_n^-(\tau) \cos n\sigma - \frac{1}{n} \frac{\mathcal{B}}{g_{11}} \beta_n^+(\tau) \sin n\sigma \tag{6}$$

$$X^2(\tau, \sigma) = x^2 + w_2 \tau + \frac{\mathcal{B}}{g_{22}} w_1 \sigma + \frac{i}{n} \beta_n^-(\tau) \cos n\sigma + \frac{1}{n} \frac{\mathcal{B}}{g_{22}} \alpha_n^+(\tau) \sin n\sigma, \tag{7}$$

where we have introduced the oscillator terms $\alpha_n^\pm(\tau) := \alpha_n(\tau) \pm \bar{\alpha}_n(\tau)$ with $\alpha_n(\tau) := \alpha_n \, e^{-in\tau}$ and $\bar{\alpha}_n(\tau) := \bar{\alpha}_n \, e^{in\tau}$ and the same definitions for β. A sum over $n > 0$ is implicit (see [6, 15] for a similar expansion).

The conjugate momenta are

$$P^1(\tau, \sigma) = \frac{1}{2\pi} \frac{\mathcal{M}}{g_{22}} \left(w_1 + \alpha_n^+ \cos n\sigma \right) \tag{8}$$

$$P^2(\tau, \sigma) = \frac{1}{2\pi} \frac{\mathcal{M}}{g_{11}} \left(w_2 + \beta_n^+ \cos n\sigma \right), \tag{9}$$

(where $\mathcal{M} := g_{11} g_{22} + \mathcal{B}^2$), and the Hamiltonian can be written as

$$H(\tau) = \frac{\pi}{2} \mathcal{M} \left[\frac{w_1^2}{g_{22}} + \frac{w_2^2}{g_{11}} + \frac{1}{g_{22}} (\alpha_n \bar{\alpha}_n + \bar{\alpha}_n \alpha_n) + \frac{1}{g_{11}} (\beta_n \bar{\beta}_n + \bar{\beta}_n \beta_n) \right]. \tag{10}$$

We are interested in commutators $[X^i(\tau, \sigma), X^j(\tau, \sigma')]$, $[X^i(\tau, \sigma), P^j(\tau, \sigma')]$, $[P^i(\tau, \sigma), P^j(\tau, \sigma')]$. It is well known [4–7] that the canonical ones are inconsistent with the new boundary conditions (5) brought about by the B_{ij} field. The simplest way of obtaining the appropriate commutators is by means of the Heisenberg equations. Comparing the series expansion for commutator $[X^i(\tau, \sigma), H(\tau)]$ with the expansion of $\dot{X}^i(\tau, \sigma)$, we obtain the commutation relations for the modes (see the appendix)

$$[x^1, w_1] = \frac{g_{22}}{\pi \mathcal{M}}, \qquad\qquad [x^2, w_2] = \frac{g_{11}}{\pi \mathcal{M}},$$
$$[\alpha_n, \bar{\alpha}_m] = -in \frac{g_{22}}{\pi \mathcal{M}} \delta_{mn}, \qquad [\beta_n, \bar{\beta}_m] = -in \frac{g_{11}}{\pi \mathcal{M}} \delta_{mn}, \tag{11}$$

which we then use to arrive at the desired commutators

$$[P^i(\sigma), P^j(\sigma')] = 0 \tag{12}$$

$$[X^i(\sigma), P^j(\sigma')] = \delta^{ij} \delta_N(\sigma - \sigma') := \frac{1}{\pi} (1 + 2 \cos n\sigma \cos n\sigma') \tag{13}$$

$$[X^i(\sigma), X^j(\sigma')] = \begin{cases} 0, & 0 < \sigma, \sigma' < \pi \\ -\dfrac{B_{ij}}{\mathcal{M}}, & \sigma = \sigma' = 0 \\ \dfrac{B_{ij}}{\mathcal{M}}, & \sigma = \sigma' = \pi \end{cases}. \tag{14}$$

So in the end, we are left with a non-commutative theory. Note that for $B_{ij} = 0$, we recover the canonical commutators.

3. *T*-duality

T-duality appears as a symmetry of closed strings when *d* coordinates of the spacetime are compactified on a torus T^d:

$$X^i \sim X^i + 2\pi m^i \tag{15}$$

with m^i being the integer numbers ($i = 1, \ldots, d$). The radii of the torus are included in the metric. We consider in this paper the case $d = 2$.

The toroidal contribution to the worldsheet action of a closed string propagating in the presence of a constant Kalb–Ramond field is the same as equation (2). The difference is that instead of boundary conditions we have periodicity of the closed string coordinates

$$X^i(\tau, \sigma + 2\pi) = X^i(\tau, \sigma) + 2\pi m^i. \tag{16}$$

The conjugate momenta are

$$P_i = \frac{1}{2\pi}(g_{ij}\dot{X}^j + B_{ij}X'^j). \tag{17}$$

The periodicity of X^i leads to a discretization of the center of mass momenta

$$p_i = \int_0^{2\pi} \mathrm{d}\sigma \, P_i = n_i, \tag{18}$$

with $n_i \epsilon \mathbf{Z}$.

The Hamiltonian corresponding to action (2) can be written as [3]

$$
\begin{aligned}
H &= \frac{1}{4\pi} \int \mathrm{d}\sigma [(2\pi)^2 P_i g^{ij} P_j + X'^i(g - Bg^{-1}B)_{ij} X'^j + 4\pi X'^i B_{ik} g^{kj} P_j] \\
&= \frac{1}{4\pi} \int \mathrm{d}\sigma \left(P_L^2 + P_R^2\right),
\end{aligned} \tag{19}
$$

where $P_L^2 = P_{La} P_{La}$ and $P_R^2 = P_{Ra} P_{Ra}$ with

$$
\begin{aligned}
P_{La} &= \tfrac{1}{\sqrt{2}}[2\pi P_i + (g - B)_{ij}X'^j]e_a^{*i} = \tfrac{1}{\sqrt{2}}g_{ij}(\dot{X}^j + X'^j)e_a^{*i} \\
P_{Ra} &= \tfrac{1}{\sqrt{2}}[2\pi P_i - (g + B)_{ij}X'^j]e_a^{*i} = \tfrac{1}{\sqrt{2}}g_{ij}(\dot{X}^j - X'^j)e_a^{*i}.
\end{aligned} \tag{20}
$$

These momenta correspond to the coordinates $\overline{X}^a = e^a{}_i X^i$ with the zweibeins defined by

$$e_i{}^a e_j{}^a = g_{ij}; \qquad e_i{}^a e_a^{*j} = \delta_i^j; \qquad e_a^{*i} e_a^{*j} = g^{ij}. \tag{21}$$

The zero-mode parts of the momenta are

$$
\begin{aligned}
p_{La} &= \tfrac{1}{\sqrt{2}}e_a^{*i}[n_i + (g - B)_{ij}m_j] \\
p_{Ra} &= \tfrac{1}{\sqrt{2}}e_a^{*i}[n_i - (g + B)_{ij}m_j],
\end{aligned} \tag{22}
$$

where n_i and m_j are the integer numbers defined in equations (16) and (18). It is convenient to write this equation in the matricial form

$$p := \begin{pmatrix} p_L \\ p_R \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^*(g - B) & e^* \\ -e^*(g + B) & e^* \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} =: VZ, \tag{23}$$

where $Z = (m_i, n_j)$ is a 4-vector composed by the winding and momentum numbers of coordinates X^1 and X^2 which are integer numbers for closed strings. Using this equation we can express the Hamiltonian as

$$H = \tfrac{1}{2}\left(p_L^2 + p_R^2\right) + N + \tilde{N} = \tfrac{1}{2}Z^t M Z + N + \tilde{N}, \tag{24}$$

where N and $\tilde{N}$ are the number operators for the oscillator modes and M is the 4×4 matrix

$$M = V^t V = \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix}.\tag{25}$$

The closed string theory has to satisfy the Virasoro constraint

$$L_0 - \tilde{L}_0 = \tfrac{1}{2}\left(\alpha_0^a \alpha_0^a - \tilde{\alpha}_0^a \tilde{\alpha}_0^a\right) + N - \tilde{N} = -\tfrac{1}{2}\left(p_L^2 - p_R^2\right) + N - \tilde{N} = 0,\tag{26}$$

where α_0^a and $\tilde{\alpha}_0^a$ are the zero modes of right and left sectors of the coordinates $\overline{X}^a$. Note that X^i are angular coordinates of the torus while $\overline{X}^a$ are the usual string coordinates expressed in string units. The Virasoro constraint can be written as

$$N - \tilde{N} = \tfrac{1}{2}\left(p_L^2 - p_R^2\right) = \tfrac{1}{2}Z^t J Z,\tag{27}$$

where J is the 4×4 matrix

$$J = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix},\tag{28}$$

with I_2 being a 2×2 identity matrix.

T-duality is a transformation of the string state and of the background that preserves the Hamiltonian (24) and the Virasoro constraint (27). This transformation acts on matrix M as

$$M \to TMT^t,\tag{29}$$

with

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\tag{30}$$

where a, b, c, d are 2×2 matrices. The invariance of the Virasoro constraint corresponds to the condition

$$TJT^t = J.\tag{31}$$

The Hamiltonian is preserved if vector Z transforms as $Z \to (T^t)^{-1}Z$ under T-duality while number operators $N, \tilde{N}$ remain unchanged. Note that for closed strings, the elements of vector Z must remain integers after this transformation. The transformation of matrix M corresponds to a change in the background g and B that can be expressed as [3]

$$E := g + B \to E^{\text{dual}} = g^{\text{dual}} + B^{\text{dual}} = (aE + b) \cdot (cE + d)^{-1}.\tag{32}$$

We are interested in particular cases of T-dualities that can be interpreted in terms of dualization of the string coordinates. First, we consider a transformation matrix of the following form:

$$T_{X^i} = \begin{pmatrix} 1 - t_{X^i} & t_{X^i} \\ t_{X^i} & 1 - t_{X^i} \end{pmatrix},\tag{33}$$

with

$$t_{X^1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; \qquad t_{X^2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.\tag{34}$$

The effect of transformation $T_{X^i} = (T_{X^i})^t = ((T_{X^i})^t)^{-1}$ on vector Z is simply to interchange the winding number and the momentum number of the corresponding coordinate: $m_i \leftrightarrow n_i$.

We now describe the effect of this transformation on the momenta. Let us choose coordinate $i = 2$. In this case, we have

$$g^{\text{dual}} = \begin{pmatrix} \frac{\mathcal{M}}{g_{22}} & \frac{\mathcal{B}}{g_{22}} \\ \frac{\mathcal{B}}{g_{22}} & \frac{1}{g_{22}} \end{pmatrix}; \qquad B^{\text{dual}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\tag{35}$$

with $\mathcal{M} := g_{11}g_{22} + \mathcal{B}^2 = \det E$. The corresponding transformed zweibeins are

$$e^{\text{dual}} = \begin{pmatrix} \sqrt{g_{11}} & \frac{\mathcal{B}}{\sqrt{g_{22}}} \\ 0 & \frac{1}{\sqrt{g_{22}}} \end{pmatrix}; \qquad e^{*\text{dual}} = \begin{pmatrix} \frac{1}{\sqrt{g_{11}}} & -\frac{\mathcal{B}}{\sqrt{g_{11}}} \\ 0 & \sqrt{g_{22}} \end{pmatrix}. \qquad (36)$$

Using these results and equation (23), we can find the transformation of momentum vector $p = (p_L, p_R)$:

$$p^{\text{dual}} = \begin{pmatrix} p_L^1 \\ p_L^2 \\ p_R^1 \\ p_R^2 \end{pmatrix}^{\text{dual}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} p_L^1 \\ p_L^2 \\ p_R^1 \\ p_R^2 \end{pmatrix} =: U_{X^2} p. \qquad (37)$$

So we see that the T-duality transformation T_{X^2} reverses the sign of the momentum component p_R^2 while preserving the sign of the other momentum components. We have introduced matrix U that represents the T-dualization of the momentum vector.

Now let us consider the case of simultaneously interchanging the winding numbers m_1, m_2 and the momentum numbers n_1, n_2. This is done by matrix

$$T_{X^1 X^2} = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \qquad (38)$$

(note that $T_{X^1 X^2} = T_{X^1} + T_{X^2}$). The transformed fields are

$$g^{\text{dual}} = \frac{1}{\mathcal{M}} \begin{pmatrix} g_{22} & 0 \\ 0 & g_{11} \end{pmatrix}; \qquad B^{\text{dual}} = \frac{1}{\mathcal{M}} \begin{pmatrix} 0 & -\mathcal{B} \\ \mathcal{B} & 0 \end{pmatrix}, \qquad (39)$$

with the corresponding zweibeins

$$e^{\text{dual}} = \frac{1}{\sqrt{\mathcal{M}}} \begin{pmatrix} \sqrt{g_{22}} & 0 \\ 0 & \sqrt{g_{11}} \end{pmatrix}; \qquad e^{*\text{dual}} = \sqrt{\mathcal{M}} \begin{pmatrix} \frac{1}{\sqrt{g_{22}}} & 0 \\ 0 & \frac{1}{\sqrt{g_{11}}} \end{pmatrix}. \qquad (40)$$

The transformation (38) in fact inverts the background matrix $E \to 1/E$. The momentum vector transforms as

$$p^{\text{dual}} = \begin{pmatrix} p_L^1 \\ p_L^2 \\ p_R^1 \\ p_R^2 \end{pmatrix}^{\text{dual}} = \frac{1}{\sqrt{\mathcal{M}}} \begin{pmatrix} \sqrt{g_{11}g_{22}} & \mathcal{B} & 0 & 0 \\ -\mathcal{B} & \sqrt{g_{11}g_{22}} & 0 & 0 \\ 0 & 0 & -\sqrt{g_{11}g_{22}} & \mathcal{B} \\ 0 & 0 & -\mathcal{B} & -\sqrt{g_{11}g_{22}} \end{pmatrix}$$

$$\times \begin{pmatrix} p_L^1 \\ p_L^2 \\ p_R^1 \\ p_R^2 \end{pmatrix} =: U_{X^1 X^2} p. \qquad (41)$$

This transformation is not as simple as that obtained in equation (37).

In the following section, we discuss concrete realizations for the T-dualities of equations (37) and (41). Before that it is useful to see how the open string parameters defined by Seiberg and Witten change under T-duality. For open strings ending on the D2-brane with mixed boundary conditions: $g_{ij} X'^j + B_{ij} \dot{X}^j = 0$, the coordinate propagator at string endpoints can be decomposed in terms of the parameters [7]

$$G^{ij} = \left(\frac{1}{E} \right)^{ij}_{\text{sym}}, \qquad \theta^{ij} = \left(\frac{1}{E} \right)^{ij}_{\text{ant}}, \qquad (42)$$

where sym(ant) means the symmetric (anti-symmetric) part. The equal time commutator is directly related to parameter θ:

$$[X^i(\tau, \sigma = 0), X^j(\tau, \sigma' = 0)] = -2\pi\theta^{ij}. \tag{43}$$

So, in principle, studying the transformation of G^{ij} and θ^{ij} after T-duality we can find out the propagator and commutator of the T-dual world.

The T_{X^2} duality transformed open string parameters are

$$G^{\text{dual}} = \left(\frac{1}{E^{\text{dual}}}\right)_{\text{sym}} = \frac{1}{g_{11}}\begin{pmatrix} 1 & -\mathcal{B} \\ -\mathcal{B} & \mathcal{M} \end{pmatrix}, \qquad \theta^{\text{dual}} = \left(\frac{1}{E^{\text{dual}}}\right)_{\text{ant}} = 0, \tag{44}$$

so we should expect a commutative dual theory. For the $T_{X^1 X^2}$ duality we find

$$G^{\text{dual}} = \begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix} = g, \qquad \theta^{\text{dual}} = \begin{pmatrix} 0 & \mathcal{B} \\ -\mathcal{B} & 0 \end{pmatrix} = B, \tag{45}$$

indicating a possible non-commutative dual theory. We will see in the following section that, although the dual θ^{ij} is different from zero, the $T_{X^1 X^2}$ transformation of the open string coordinates is such that a non-commutative primal system turns into a commutative one.

4. Duality transformations of open string coordinates

From now on, we take a coordinate-focused approach to T-duality. That is, we will be speaking of 'dualizing' the fields $X^i(\tau, \sigma)$, though still keeping in mind that T-duality is a symmetry of the Hamiltonian, and acts upon the momenta. This will enable us to assess the commutative/non-commutative character of the theories connected by T-duality.

We will discuss here two possible ways of dualizing the string coordinates in the presence of a constant Kalb–Ramond field. The first one is the T-duality mechanism that consists in introducing a Lagrange multiplier in the action which is eventually identified with the dual coordinate(s). By rewriting the action in terms of the dual coordinate, we get the transformations for the background. This approach was developed by Buscher [13].

The other mechanism is an extrapolation of the $\mathcal{B} = 0$ T-duality prescription: we simply interchange the τ and σ derivatives of the coordinate(s) X^i to be dualized. This corresponds to inverting the sign of the right component of the string coordinates $\left(X_R^i \to -X_R^i\right)$. As we shall see, although this alternative mechanism preserves the Hamiltonian and the Virasoro constraint in the same way as T-duality, they are not equivalent.

We consider two cases:

A. Dualizing one coordinate: X^2,
B. Dualizing both coordinates X^1 and X^2.

To each case, we apply both mechanisms mentioned above.

We start from a D2-brane with a constant Kalb–Ramond field wrapping the torus. This primal system is non-commutative, as discussed in section 2, as a consequence of the mixed boundary conditions at the string endpoints:

$$g_{11}X'^1 + \mathcal{B}\dot{X}^2 = 0, \qquad g_{22}X'^2 - \mathcal{B}\dot{X}^1 = 0. \tag{46}$$

Other interesting consequence of these BCs is that the 'winding' and momentum numbers of open string coordinates X^1 and X^2 are now related:

$$m^1 = -\frac{\mathcal{B}}{\mathcal{M}}n^2 m^2 = \frac{\mathcal{B}}{\mathcal{M}}n^1, \tag{47}$$

so the primal 'winding' numbers are not integer numbers, unlike the closed string case.

In the following subsections, we will discuss the dualization of open string coordinates, backgrounds and boundary conditions and study its effect on non-commutativity.

4.1. Dualizing one coordinate

4.1.1. T-duality transformation. Let us begin by defining the worldsheet vector

$$v_\alpha^2 := \partial_\alpha X^2, \tag{48}$$

where $\alpha = \tau, \sigma$. Action (2) becomes

$$S = \frac{1}{4\pi} \int d\tau \, d\sigma \big[-\sqrt{h} h^{\alpha\beta} \big(g_{11} \partial_\alpha X^1 \partial_\beta X^1 + g_{22} v_\alpha^2 v_\beta^2 \big) + 2\epsilon^{\alpha\beta} \mathcal{B} \partial_\alpha X^1 v_\beta^2 \big]. \tag{49}$$

Now we add the (vanishing) Lagrange multiplier:

$$S \rightarrow S - \frac{1}{2\pi} \int d\tau \, d\sigma \epsilon^{\alpha\beta} \partial_\alpha X_S^2 v_\beta^2. \tag{50}$$

If we vary this action with respect to the new coordinate X_S^2, we recover the primal action (2) when using (48). If, instead, we vary with respect to v_α^2, we find the following equation of motion:

$$v_\alpha^2 = -\frac{1}{g_{22}} \epsilon_\alpha^\beta \partial_\beta \big[X_S^2 + \mathcal{B} X^1 \big]. \tag{51}$$

Substituting this equation into (50) we find the 'dual' action,

$$S^S = \frac{1}{4\pi} \int d\tau \, d\sigma \big[-h^{\alpha\beta} g_{ij}^S \partial_\alpha X_S^i \partial_\beta X_S^j + \epsilon^{\alpha\beta} B_{ij}^S \partial_\alpha X_S^i \partial_\beta X_S^j \big], \tag{52}$$

where dual fields g_{ij}^S and B_{ij}^S are precisely the same found in equation (35). Thus, we note that this coordinate transformation indeed represents a realization of the *T*-duality studied in section 3. Note that g_{ij}^S is non-diagonal.

Consistency between equations (48) and (51) yields the relations between the dual coordinate $X_S^2(\tau, \sigma)$ and the primal one:

$$\dot{X}_S^2 = g_{22} X'^2 - \mathcal{B} \dot{X}^1 \qquad X_S'^2 = g_{22} \dot{X}^2 - \mathcal{B} X'^1. \tag{53}$$

In terms of coordinates $\overline{X}^a = e^a_{\ i} X^i$ these transformations read

$$\dot{\overline{X}}_S^2 = \sqrt{g_{22}^S} \dot{X}_S^2 = \overline{X}'^2 - \overline{\mathcal{B}} \dot{\overline{X}}^1, \qquad \overline{X}_S'^2 = \sqrt{g_{22}^S} X_S'^2 = \dot{\overline{X}}^2 - \overline{\mathcal{B}} \overline{X}'^1, \tag{54}$$

where $\overline{\mathcal{B}} := \mathcal{B}/\sqrt{g_{11} g_{22}}$. Note that in the case $\mathcal{B} = 0$, this *T*-duality transformation corresponds to inter-changing the τ and σ derivatives for the $\overline{X}^2$ coordinate.

Now we use equations (20), (35) and (53) to check the *T*-duality transformation of the momentum vector. We find

$$P_S = \begin{pmatrix} P_{SL}^1 \\ P_{SL}^2 \\ P_{SR}^1 \\ P_{SR}^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} P_L^1 \\ P_L^2 \\ P_R^1 \\ P_R^2 \end{pmatrix} = U_{X^2} P. \tag{55}$$

This result is consistent with the transformation of the zero-mode momentum vector given in (37) and confirms that the Hamiltonian is preserved.

What about non-commutativity? Using the primal D2-brane boundary conditions (46) and relations (53), we find the dual boundary conditions

$$X'^1 + \frac{\mathcal{B}}{\mathcal{M}} X_S'^2 = 0, \qquad \dot{X}_S^2 = 0, \tag{56}$$

so that X_S^2 satisfies Dirichlet boundary conditions, and the other BC is some kind of rotated Neumann condition. Hence, our former non-commutativity at the string endpoints is lost,

and we are left with a commutative dual system! This result can be checked by computing the commutators of the string coordinate operators following a procedure similar to that of section 2 (but with a zero Kalb–Ramond field and a non-diagonal metric). Below, we will give a geometrical picture of this *T*-dual system in terms of a tilted (and non-localized) D1-brane. For an interesting discussion of this system see [16].

4.1.2. Alternative transformation. Let us now consider the other, more direct means of producing dual coordinates. Define the alternative dual coordinate X_A^2 by

$$\dot{X}_A^2 := g_{22} X'^2, \qquad X_A'^2 := g_{22} \dot{X}^2 \tag{57}$$

and

$$g^A := \begin{pmatrix} g_{11} & 0 \\ 0 & \frac{1}{g_{22}} \end{pmatrix}. \tag{58}$$

This simple operation leads to the same momentum vector transformation (55). So the Hamiltonian (19) is preserved. In terms of the planar coordinates $\overline{X}^a = e^a{}_i X^i$ these transformations read

$$\dot{\overline{X}}_A^2 = \sqrt{g_{22}^A} \dot{X}_A^2 = \overline{X}'^2 \qquad \overline{X}_A'^2 = \sqrt{g_{22}^A} X_A'^2 = \dot{\overline{X}}^2. \tag{59}$$

Note that for $\mathcal{B} = 0$, these transformations are the same as those given in the standard mechanism (54).

Using the primal D2-brane boundary conditions and relations (57), we find the dual ones

$$g_{11} X'^1 + \frac{\mathcal{B}}{g_{22}} X_A'^2 = 0, \qquad \dot{X}_A^2 - \mathcal{B}\dot{X}^1 = 0, \tag{60}$$

for $\sigma = 0, \pi$. These BCs can be rewritten as

$$\overline{X}'^1 + \overline{\mathcal{B}}\overline{X}_A'^2 = 0, \qquad \dot{\overline{X}}_A^2 - \overline{\mathcal{B}}\dot{\overline{X}}^1 = 0. \tag{61}$$

Therefore, $\overline{X}^1$ and $\overline{X}_A^2$ are nothing more than rotations of plain Neumann and Dirichlet coordinates $\overline{Y}^1$ and $\overline{Y}^2$ defined by

$$\begin{pmatrix} \overline{Y}^1 \\ \overline{Y}^2 \end{pmatrix} := (\overline{\mathcal{M}})^{-1/2} \begin{pmatrix} 1 & \overline{\mathcal{B}} \\ -\overline{\mathcal{B}} & 1 \end{pmatrix} \begin{pmatrix} \overline{X}^1 \\ \overline{X}_A^2 \end{pmatrix} \tag{62}$$

where $\overline{\mathcal{M}} := \frac{\mathcal{M}}{g_{11}g_{22}}$. Coordinates $\overline{Y}^1$ and $\overline{Y}^2$ are typical of a D1-brane so our results tell us that the dual world consists of a non-localized tilted D1-brane (with a zero Kalb–Ramond field) that corresponds to a commutative dual system. We can construct an action for X_A^2 of type

$$S^A = \frac{1}{4\pi} \int d\tau \, d\sigma \left[-\sqrt{h} h^{\alpha\beta} g_{ij}^A \partial_\alpha X_A^i \partial_\beta X_A^j + \epsilon^{\alpha\beta} B_{ij}^A \partial_\alpha X_A^i \partial_\beta X_A^j \right], \tag{63}$$

with g_{ij}^A given by equation (58). It is straightforward to show that the dual boundary conditions (60) force us to define $B_{ij}^A = 0$ which confirms the fact that the dual theory is commutative.

Finally, we can find a connection between the *T*-duality transformation and the alternative transformation. Using equations (53) and (57) we get

$$X_S^2 = X_A^2 - \mathcal{B}X^1, \tag{64}$$

which relates the dual coordinates X_S^2 and X_A^2. It is straightforward to show that X_S^2 is proportional to the Dirichlet coordinate $\overline{Y}_2$ transversal to the tilted D1-brane. Using (64) we can see that the actions S^A and S^S of equations (52) and (63) are equivalent. These

two mechanisms for the dualization of one coordinate seem to lead to the same dual world. However, if we calculate the matrix $T_{X^2}^A$,

$$T_{X^2}^A = \begin{pmatrix} 1 & 0 & 0 & -\mathcal{B} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & \mathcal{B} & 0 \end{pmatrix} \neq T_{X^2}^S = T_{X^2}, \tag{65}$$

we see that the winding and momentum numbers transform differently from the standard T-duality. Although we consider open strings, we note that applying $T_{X^2}^A$ to closed strings leads in general to non-integer momentum and winding numbers. This means that this alternative transformation is not a T-duality transformation. The particular cases of $\mathcal{B}$ having integer values would be exceptions.

4.2. Dualizing two coordinates

4.2.1. T-duality transformation.

Now we introduce two worldsheet vectors

$$v_\alpha^1 := \partial_\alpha X^1, \qquad v_\alpha^2 := \partial_\alpha X^2. \tag{66}$$

Then the action analogous to (50) is

$$S = \frac{1}{4\pi} \int d^2\sigma \left[-\sqrt{-h} h^{\alpha\beta} \left(g_{11} v_\alpha^1 v_\beta^1 + g_{22} v_\alpha^2 v_\beta^2 \right) \right.$$
$$\left. + 2\epsilon^{\alpha\beta} \mathcal{B} v_\alpha^1 v_\beta^2 - 2 X_S^1 \epsilon^{\alpha\beta} \partial_\alpha v_\beta^1 - 2 X_S^2 \epsilon^{\alpha\beta} \partial_\alpha v_\beta^2 \right]. \tag{67}$$

We calculate the equation of motion for each v_α^i:

$$v_\alpha^1 = \frac{g_{22}}{\mathcal{M}} \partial_\rho \left(\epsilon_\alpha^\rho X_S^1 + \frac{\mathcal{B}}{g_{22}} \delta_\alpha^\rho X_S^2 \right)$$
$$v_\alpha^2 = \frac{g_{11}}{\mathcal{M}} \partial_\rho \left(\epsilon_\alpha^\rho X_S^2 - \frac{\mathcal{B}}{g_{11}} \delta_\alpha^\rho X_S^2 \right), \tag{68}$$

and substitute back in the action, to obtain the dual action:

$$S^S = \frac{1}{2} \int d^2\sigma \, h^{\alpha\beta} \left(g_{11}^S \partial X_S^1 \partial_\beta X_S^2 + g_{22}^S \partial X_S^2 \partial_\beta X_S^2 \right) + 2\epsilon^{\alpha\beta} B_{12}^S \partial_\alpha X_S^1 \partial_\beta X_S^2, \tag{69}$$

with $g_{11}^S = \frac{g_{22}}{\mathcal{M}}, g_{22}^S = \frac{g_{11}}{\mathcal{M}}, B_{12}^S = -\frac{\mathcal{B}}{\mathcal{M}}$. The background matrix, then, is inverted:

$$E^S = \begin{pmatrix} g_{11}^S & B_{12}^S \\ -B_{12}^S & g_{22}^S \end{pmatrix} = \frac{1}{\mathcal{M}} \begin{pmatrix} g_{22} & -\mathcal{B} \\ \mathcal{B} & g_{11} \end{pmatrix} = \frac{1}{E}. \tag{70}$$

From equations (66) and (68), we find the relation between primal and dual coordinates

$$\partial_\alpha X_S^1 = \left(g_{11} \epsilon_\alpha^\rho \partial_\rho X^1 + \mathcal{B} \partial_\alpha X^2 \right), \qquad \partial_\alpha X_S^2 = \left(g_{22} \epsilon_\alpha^\rho \partial_\rho X^2 - \mathcal{B} \partial_\alpha X^1 \right). \tag{71}$$

Using these relations in the primal boundary conditions (46), we find the dual boundary conditions

$$\dot{X}_S^1 = g_{11} X'^1 + \mathcal{B} \dot{X}^2 = 0, \qquad \dot{X}_S^2 = g_{22} X'^2 - \mathcal{B} \dot{X}^1 = 0. \tag{72}$$

Thus both dual coordinates are of the Dirichlet type. Consequently, the dual system is a D0-brane and we have commutativity!

Using the dual zweibein

$$e_S^* = \sqrt{\mathcal{M}} \begin{pmatrix} \frac{1}{\sqrt{g_{22}}} & 0 \\ 0 & \frac{1}{\sqrt{g_{11}}} \end{pmatrix} \tag{73}$$

and equations (70) and (71), we find the matrix $U^S_{X^1X^2}$ that transforms the momentum vector

$$
P_S = \begin{pmatrix} P^1_{SL} \\ P^2_{SL} \\ P^1_{SR} \\ P^2_{SR} \end{pmatrix} = \frac{1}{\mathcal{M}} \begin{pmatrix} \sqrt{g_{11}g_{22}} & \mathcal{B} & 0 & 0 \\ -\mathcal{B} & \sqrt{g_{11}g_{22}} & 0 & 0 \\ 0 & 0 & -\sqrt{g_{11}g_{22}} & \mathcal{B} \\ 0 & 0 & -\mathcal{B} & -\sqrt{g_{11}g_{22}} \end{pmatrix} \begin{pmatrix} P^1_L \\ P^2_L \\ P^1_R \\ P^2_R \end{pmatrix} = U^S_{X^1X^2} P
$$
(74)

which is the same found in section 3 for the zero-mode momentum vector.

4.2.2. Alternative transformation. In this case, the dual coordinates X^i_A are introduced by

$$
\dot{X}^i_A(\tau,\sigma) := g_{ij} X'^{j}(\tau,\sigma), \qquad X'^{i}_A(\tau,\sigma) := g_{ij}\dot{X}^j(\tau,\sigma),
$$
(75)

where g_{ij} is the primal diagonal metric. The dual boundary conditions,

$$
\frac{1}{g_{11}} X'^1_A - \frac{1}{\mathcal{B}} \dot{X}^2_A = 0, \qquad \frac{1}{g_{22}} X'^2_A + \frac{1}{\mathcal{B}} \dot{X}^1_A = 0,
$$
(76)

have the same mixed form of those of the primal system. This shows that the dual system is a D2-brane with a nonzero Kalb–Ramond field that has non-commutative behavior. From these BCs, we figure out the dual background

$$
g^A = g^{-1}, \qquad B^A = \begin{pmatrix} 0 & -\frac{1}{\mathcal{B}} \\ \frac{1}{\mathcal{B}} & 0 \end{pmatrix}.
$$
(77)

Note that the zweibein has inverted, too. We proceed to the transformation of the momenta:

$$
P_A = \begin{pmatrix} P^1_{AL} \\ P^2_{AL} \\ P^1_{AR} \\ P^2_{AR} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} P^1_L \\ P^2_L \\ P^1_R \\ P^2_R \end{pmatrix} =: U^A_{X^1X^2} P.
$$
(78)

This transformation is clearly a symmetry of the Hamiltonian, and it also preserves $P^A_L - P^2_R$. This time, though, the momentum vector transformation $U^A_{X^1X^2}$ is not the same as that obtained using T-duality $\left(U^S_{X^1X^2}\right)$. Moreover, the winding–momentum number transformation $T^A_{X^1X^2}$ is also different,

$$
T^A_{X^1X^2} = \begin{pmatrix} 0 & -\frac{1}{\mathcal{B}} & 0 & 0 \\ \frac{1}{\mathcal{B}} & 0 & 0 & 0 \\ 1 & 0 & 0 & -\mathcal{B} \\ 0 & 1 & \mathcal{B} & 0 \end{pmatrix} \neq T^S_{X^1X^2} = T_{X^1X^2},
$$
(79)

where $T_{X^1X^2}$ is as defined in equation (38). Again we note that the matrix $T^A_{X^1X^2}$ applied to closed strings leads in general to non-integer momentum and winding numbers. So, as already pointed out in the case of transforming only one coordinate, the alternative transformation is not a T-duality (the $\mathcal{B} = 1$ case would be an exception).

Regarding commutativity of position operators, while the dual system of the X^i_S coordinates and E^S background is a commutative one, the dual system of the X^i_A coordinates is non-commutative. Indeed, according to (14) and (77), we have

$$
\left[X^1_A(\tau,0), X^2_A(\tau,0) \right] = -\frac{B^A_{12}}{\mathcal{M}^A} = \frac{\frac{1}{\mathcal{B}}}{\frac{\mathcal{M}}{g_{11}g_{22}\mathcal{B}^2}} = \frac{\mathcal{B}g_{11}g_{22}}{\mathcal{M}} = -g_{11}g_{22}[X^1(\tau,0), X^2(\tau,0)].
$$
(80)

That means that the dual commutator is proportional to the primal one.

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5. Conclusions

We have studied the effect of *T*-duality in non-commutativity for open string coordinates in the presence of a Kalb–Ramond antisymmetric background. We discussed the fact that the transformation of the background (metric and Kalb–Ramond field) concerns only the behavior of the target space. The transformation of a particular system living in the target space is defined by the transformation of the boundary conditions. We considered as our primal system a D2-brane wrapped on a T^2 torus (target space). This system has mixed boundary conditions and is a two-dimensional non-commutative space.

Considering the *T*-dualization of just one coordinate, we found a commutative dual system. This can be understood from the fact that *T*-duality of one coordinate transforms the original D2-brane into a (non-localized) tilted D1-brane. On the other hand, *T*-duality transforms the target space into another T^2 torus without a Kalb–Ramond field. The alternative transformation applied to one coordinate leads to an equivalent commutative system.

When *T*-dualizing both coordinates, we found a commutative system since the dual boundary conditions are all of the Dirichlet type, indicating that the dual system is a (non-localized) D0-brane. This is a non-trivial result since the dual target space is a T^2 torus with a non-vanishing Kalb–Ramond field. It is important to remark that a different system, like a D2-brane, living in this dual target space will be non-commutative. The commutative/non-commutative character of open strings depends not only on the target space but also on the boundary conditions, which define a particular D-brane system. Even in the primal target space, the presence of a Kalb–Ramond field does not rule out the possibility of a commutative system, like a D0-brane.

On the other hand, the alternative transformation applied to two coordinates leads to a dual system with mixed boundary conditions, corresponding to a non-commutative D2-brane which is not equivalent to the D0-system obtained by *T*-duality. We remark that the alternative transformation is not a *T*-duality since it does not preserve the condition that, for closed strings, winding and momentum numbers (in the compact directions) are integer numbers.

It may be surprising that non-commutativity is lost for some *T*-duality transformations, but we must remember that the *T*-duality transformation acts only on the compact coordinates $X^i, i = 1, 2$. The non-compact coordinates $X^I, I = 3, 4, \ldots$, have their commutation relations unchanged. Our non-commutative parameter lives on a torus. This situation differs from the case of non-commutative quantum field theories formulated in non-compact spaces where the non-commutativity parameter is taken as a physical quantity. Since we expect *T*-duality transformation to be a symmetry of open–closed string theory, the non-commutativity parameter of the compact dimensions should not be a physical observable.

Acknowledgments

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